Fractional dynamical systems and applications in mechanics and economics

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Abstract

Using the fractional integration and differentiation on $\mathbb{R}$ we build the fractional jet fibre bundle on a differentiable manifold and we emphasize some important geometrical objects. Euler-Lagrange fractional equations are described. Some significant examples from mechanics and economics are presented.

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1 Introduction

The operators of fractional differentiation have been introduced by Leibnitz, Liouville, Riemann, Grunwal and Letnikov [6]. The fractional derivatives and integrals are used in the description of some models in mechanics, physics [6], economics [4] and medicine [11]. The fractional variational calculus [1] is an important instrument in the analysis of such models. The Euler-Lagrange equations are non-autonomous fractional differential equations in those models.

In this paper we present the fractional jet fibre bundle of order k on a differentiable manifold as being $J^{\alpha k}(\mathbb{R}, M) = \mathbb{R} \times Osc^{\alpha k}(M)$, $\alpha \in (0, 1)$, $k \in \mathbb{N}^*$.

The fibre bundle $J^{\alpha k}$ is built in a similar way as the fibre bundle E^k by R. Miron [9]. Among the geometrical structures defined on $J^\alpha(\mathbb{R}, M)$ we consider the dynamical fractional connection and the fractional Euler-Lagrange equations associated with a function defined on $J^{\alpha k}(\mathbb{R}, M)$.

In section 2 we describe the fractional operators on $\mathbb{R}$ and some of their properties which are used in the paper. In section 3 we describe the fractional osculator bundle of order k. In section 4 the fractional jet fibre bundle $J^\alpha(\mathbb{R}, M)$ is defined, the fractional dynamical connection is built and the fractional Euler-Lagrange equations are established using the notion of fractional extremal value and classical extremal value on $J^{\alpha k}(\mathbb{R}, M)$. In section 5 we consider some examples and applications.

2 Elements of fractional integration and differentiation on $\mathbb{R}$

Let $f : [a, b] \to \mathbb{R}$ be an integrable function and $\alpha \in (0, 1)$. The left-sided (right-sided) fractional derivative of f is the function

$$\begin{aligned} ({}_-D_t^\alpha f)(t) &= \tfrac{1}{\Gamma(1-\alpha)} \tfrac{d}{dt} \int_a^t \tfrac{f(s)-f(a)}{(t-s)^\alpha} ds \\ ({}_+D_t^\alpha f)(t) &= \tfrac{1}{\Gamma(1-\alpha)} \tfrac{d}{dt} \int_t^b \tfrac{f(b)-f(s)}{(s-t)^\alpha} ds, \end{aligned} \tag{1}$$

where $t \in [a,\ b)$ and Γ is Euler's gamma function.

Proposition 1. (see [6]) The operators ${}_-D_t^\alpha$ and ${}_+D_t^\alpha$ have the properties:
1. If f_1 and f_2 are defined on $[a, b]$ and ${}_-D_t^\alpha$, ${}_+D_t^\alpha$ exists, then

$${}_-D_t^\alpha(c_1 f_1 + c_2 f_2)(t) = c_1({}_-D_t^\alpha f_1)(t) + c_2({}_-D_t^\alpha f_2)(t). \tag{2}$$

2. If $\{\alpha_n\}_{n\geq 0}$ is a real number sequence with $\lim\limits_{n\to\infty} \alpha_n = 1$ then

$$\lim_{n\to\infty}({}_-D_t^{\alpha_n} f)(t) = ({}_-D_t^1 f)(t) = \frac{d}{dt} f(t). \tag{3}$$

3. a) If $f(t) = c$, $t \in [a, b]$, $c \in \mathbb{R}$ then

$$({}_-D_t^\alpha f)(t) = 0. \tag{4}$$

b) If $f(t) = t^\gamma$, $t \in (a, b]$, $\gamma \in \mathbb{R}$, then

$$({}_-D_t^\alpha f)(t) = \frac{t^{\gamma-\alpha}\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}. \tag{5}$$

c) If $f(t)=\frac{t^{\alpha}}{\Gamma(1+\alpha)}$, then

$$({}_{-}D_t^{\alpha}f)(t)=1. \tag{6}$$

4. If f_1 and f_2 are analytic functions on $[a,b]$ then

$$({}_{-}D_t^{\alpha}(f_1f_2))(t)=\sum_{k=0}^{\infty}\left(\begin{array}{c}\alpha\\ k\end{array}\right)({}_{-}D_t^{\alpha-k}f_1)(t)\frac{d^k}{(dt)^k}f_2(t), \tag{7}$$

where $\frac{d^k}{(dt)^k}=\frac{d}{dt}\circ\frac{d}{dt}\circ\ldots\circ\frac{d}{dt}$.
5. It also holds true

$$\int_a^b f_1(t)({}_{-}D_t^{\alpha}f_2)(t)dt=-\int_a^b f_2(t)({}_{+}D_t^{\alpha}f_1)(t)dt. \tag{8}$$

6. a) If $f:[a,b]\to\mathbb{R}$ admits fractional derivatives of order $a\alpha$, $a\in\mathbb{N}$, then

$$f(t+h)=E_{\alpha}((ht)^{\alpha}{}_{-}D_t^{\alpha})f(t), \tag{9}$$

where E_{α} is the Mittag-Leffler function given by

$$E_{\alpha}(t)=\sum_{a=0}^{\infty}\frac{t^{\alpha a}}{\Gamma(1+\alpha a)}. \tag{10}$$

b) If $f:[a,b]\to\mathbb{R}$ is analytic and $0\in(a,b)$ then the fractional McLaurin series is

$$f(t)=\sum_{a=0}^{\infty}\frac{t^{\alpha a}}{\Gamma(1+\alpha a)}({}_{-}D_t^{\alpha a}f)(t)\mid_{t=0}. \tag{11}$$

The physical and geometrical interpretation of the fractional derivative on $\mathbb{R}$ is suggested by the interpretation of the Stieltjes integral, because the integral used in the definition of the fractional derivative is a Riemann-Stieltjes integral [10].

By definition, the left-sided (right-sided) fractional derivative of f, of order α, $m=[\alpha]+1$, is the function

$$\begin{array}{l} D_t^{\alpha}f(t)=\frac{1}{\Gamma(m-\alpha)}\left(\frac{d}{dt}\right)^m\int_{-\infty}^t\frac{f(s)-f(0)}{(t-s)^{\alpha}}ds,\ 0\in(-\infty,t)\\ {}^*D_t^{\alpha}f(t)=\frac{1}{\Gamma(m-\alpha)}\left(-\frac{d}{dt}\right)^m\int_t^{\infty}\frac{f(s)-f(0)}{(s-t)^{\alpha}}ds,\ 0\in(t,\infty).\end{array} \tag{12}$$

If $\overline{\mathrm{supp}f}\subset[a,b]$, then $D_t^{\alpha}f={}_{-}D_t^{\alpha}f$, ${}^*D_t^{\alpha}f={}_{+}D_t^{\alpha}f$.

Let us consider the seminorms

$$\begin{array}{l}|x|_{J_L^{\alpha}(\mathbb{R})}=\|D_t^{\alpha}x\|_{L^2(\mathbb{R})}\\ |x|_{J_R^{\alpha}(\mathbb{R})}=\|{}^*D_t^{\alpha}x\|_{L^2(\mathbb{R})},\end{array}$$

and the norms

$$\|x\|_{J_L^\alpha(\mathbb{R})} = \left(\|x\|^2_{L^2(\mathbb{R})} + |x|^2_{J_L^\alpha(\mathbb{R})}\right)^{1/2}$$
$$\|x\|_{J_R^\alpha(\mathbb{R})} = \left(\|x\|^2_{L^2(\mathbb{R})} + |x|^2_{J_R^\alpha(\mathbb{R})}\right)^{1/2},$$

and $J^\alpha_{0L}(\mathbb{R})$, $J^\alpha_{0R}(\mathbb{R})$ the closures of $C_0^\infty(\mathbb{R})$ with respect to the two norms from above, respectively. In [6] it is proved that the operators D_t^α and ${}^*D_t^\alpha$ satisfy the properties:

Proposition 2. *Let $I \subset \mathbb{R}$ and let $J^\alpha_{0L}(I)$ and $J^\alpha_{0R}(I)$ be the closures of $C_0^\infty(I)$ with respect to the norms from above. For any $x \in J^\beta_{0L}(I)$, $0 < \alpha < \beta$, the following relation holds:*

$$D_t^\beta x(t) = D_t^\alpha D_t^{\beta-\alpha} x(t).$$

For any $x \in J^\beta_{0R}(I)$, $0 < \alpha < \beta$, it also holds

$${}^*D_t^\beta x(t) = {}^*D_t^{\alpha *} D_t^{\beta-\alpha} x(t).$$

In the following we shall consider the fractional derivatives defined above.

3 The fractional osculator bundle of order k on a differentiable manifold

Let $\alpha \in (0,1]$ be fixed and M a differentiable manifold of dimension n. Two curves $\rho,\ \sigma : I \to \mathbb{R}$, with $\rho(0) = \sigma(0) = x_0 \in M$, $0 \in I$, have a fractional contact α of order $k \in \mathbb{N}^*$ in x_0, if for any $f \in \mathcal{F}(U)$, $x_0 \in U$, U a chart on M, it holds

$$D_t^{\alpha a}(f \circ \rho)\,|_{t=0} = D_t^{\alpha a}(f \circ \sigma)\,|_{t=0} \tag{13}$$

where $a = \overline{1,\ k}$. The relation (13) is an equivalence relation. The equivalence class $[\rho]^{\alpha k}_{x_0}$ is called the fractional k-osculator space of M in x_0 and it will be denoted by $Osc^{\alpha k}_{x_0}(M)$. If the curve $\rho : I \to M$ is given by $x^i = x^i(t)$, $t \in I$, $i = \overline{1,\ n}$, then, considering the formula (11), the class $[\rho]^{\alpha k}_{x_0}$, may be written as

$$x^i(t) = x^i(0) + \frac{t^\alpha}{\Gamma(1+\alpha)} D_t^\alpha x^i(t)\,|_{t=0} + ... + \frac{t^{\alpha k}}{\Gamma(1+\alpha k)} D_t^{\alpha k} x^i(t)\,|_{t=0}\ , \tag{14}$$

where $t \in (-\varepsilon, \varepsilon)$. We shall use the notation

$$x^i(0) = x^i, \quad y^{i(\alpha a)} = \frac{1}{\Gamma(1+\alpha a)} D_t^{\alpha a} x^i(t)\,|_{t=0}\ , \tag{15}$$

for $i=\overline{1,n}$ and $a=\overline{1,k}$.
By definition, the fractional osculator bundle of order r is the fibre bundle $(Osc^{\alpha k}(M), M)$ where $Osc^{\alpha k}(M)=\bigcup_{x_0\in M} Osc^{\alpha k}_{x_0}(M)$ and
$\pi_0^{\alpha k}: Osc^{\alpha k}(M)\to M$ is defined by $\pi_0^{\alpha k}([\rho]^{\alpha k}_{x_0})=x_0$, $(\forall)[\rho]^{\alpha k}_{x_0}\in Osc^{\alpha k}(M)$.

For $f\in\mathcal{F}(U)$, the fractional derivative of order α, $\alpha\in(0,\ 1)$, with respect to the variable x^i, is defined by

$$(D^\alpha_{x^i}f)(x)= \\ \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial x^i}\int_{a^i}^{x^i}\frac{f(x^1,...,x^{i-1},s,x^{i+1},...,x^n)-f(x^1,...,x^{i-1},a^i,x^{i+1},...,x^n)}{(x^i-s)^\alpha}ds, \tag{16}$$

where x^i are the coordinate functions on U, $\frac{\partial}{\partial x^i}$, $i=\overline{1,n}$, is the canonical base of the vector fields on U and $U_{ab}=\{x\in U,\ a^i\le x^i\le b^i,\ i=\overline{1,n}\}\subset U$. Let $U,\ U'\subset M$ be two charts on M, $U\cap U'\neq\emptyset$ and consider the change of variable

$$\bar{x}^i=\bar{x}^i(x^1,...,x^n) \tag{17}$$

with $\det\left(\frac{\partial\bar{x}^i}{\partial x^j}\right)\neq 0$. Let $\{dx^i\}_{i=\overline{1,\ n}}$ be the canonical base of 1-forms of $\mathcal{D}^1(U)$ and let us define the 1-forms $d(x^i)^\alpha=\alpha(x^i)^{\alpha-1}dx^i$, $i=\overline{1,n}$. The exterior differential $d^\alpha:\mathcal{F}(U\cap U')\to\mathcal{D}^1(U\cap U')$ is defined by

$$d^\alpha=d(x^j)^\alpha D^\alpha_{x^j}=d(\bar{x}^j)^\alpha D^\alpha_{\bar{x}^j}. \tag{18}$$

Using (18) and the property $D^\alpha_{x^i}\left(\frac{(x^i)^\alpha}{\Gamma(1+\alpha)}\right)=1$, it follows that

$$d(x^j)^\alpha=\frac{1}{\Gamma(1+\alpha)}D^\alpha_{\bar{x}^i}(x^j)^\alpha d(\bar{x}^i)^\alpha. \tag{19}$$

Using the notation

$$\overset{\alpha}{J}{}^j_i(x,\ \bar{x})=\frac{1}{\Gamma(1+\alpha)}D^\alpha_{\bar{x}^i}(x^j)^\alpha, \tag{20}$$

from (19) we get

$$d(x^j)^\alpha=\overset{\alpha}{J}{}^j_i(x,\ \bar{x})d(\bar{x}^i)^\alpha. \tag{21}$$

From (21) it follows that

$$\overset{\alpha}{J}{}^j_i(x,\ \bar{x})\,\overset{\alpha}{J}{}^i_h(x,\ \bar{x})=\delta^j_h. \tag{22}$$

Consider $x^i=x^i(t)$ and $\bar{x}^i(t)=\bar{x}^i(x(t))$, $i=\overline{1,n}$, $t\in I$. Applying the operator D^α_t we get

$$(D^\alpha_t\bar{x}^i)(t)=D^\alpha_{x^j}\bar{x}^i(x)(D^\alpha_t x^j)(t)=\overset{\alpha}{J}{}^i_j(\bar{x},\ x)(D^\alpha_t x^j)(t). \tag{23}$$

Considering the notation from (15) we have

$$y^{i(\alpha)} = \overset{\alpha}{J}{}^i_j(\bar{x},\ x)\bar{y}^{j(\alpha)}. \tag{24}$$

Also, from (15) we deduce

$$D_t^\alpha y^{i(\alpha a)} = \frac{\Gamma(\alpha a)}{\Gamma(\alpha(a-1))} y^{i(\alpha a)}, \tag{25}$$

where $i = \overline{1,n}$. Applying the operator D_t^α in the relation (24) we find

$$\begin{aligned} &\tfrac{\Gamma(\alpha(a-1))}{\Gamma(\alpha)}\bar{y}^{i(\alpha a)} = \Gamma(1+\alpha)\,\overset{\alpha}{J}{}^i_j(\bar{y}^{\alpha(a-1)},\ x)y^{j(\alpha)} + \\ &\tfrac{\Gamma(2\alpha)}{\Gamma(\alpha)}\,\overset{\alpha}{J}{}^i_j(y^{(\alpha(a-1))},\ y^\alpha)y^{j(2\alpha)} + ... + \tfrac{\Gamma(2\alpha)}{\Gamma(\alpha)}\,\overset{\alpha}{J}{}^i_j(\bar{y}^{\alpha(a-1)},\ y^{\alpha b})y^{j((b+1)\alpha)} + \\ &... + \tfrac{\Gamma(\alpha(a-1))}{\Gamma(\alpha)} y^{i(\alpha a)}, \end{aligned} \tag{26}$$

where $a = \overline{1,k}$.

Proposition 3. (see [2], [5])
a) The coordinate transformation on $Osc^{(\alpha k)}(M)$,
$(x^i, y^{i(\alpha)}, ..., y^{i(\alpha k)}) \to (\bar{x}^i, \bar{y}^{i(\alpha)}, ..., \bar{y}^{i(\alpha k)})$ are given by the formulas (17) and (26).
b) The operators $D^\alpha_{x^i}$ and the 1-forms $(dx^i)^\alpha$, $i = \overline{1,n}$, transform by the formulas

$$\begin{aligned} D^\alpha_{\bar{x}^i} &= \overset{\alpha}{J}{}^i_j(x,\ \bar{x})D^\alpha_{x^j} \\ d(\bar{x}^i)^\alpha &= \overset{\alpha}{J}{}^i_j(\bar{x},\ x)d(x^j)^\alpha. \end{aligned} \tag{27}$$

4 The fractional jet bundle of order k on a differentiable manifold; geometrical objects

By definition, the k-order fractional jet bundle is the space $J^{\alpha k}(\mathbb{R},\ M) = \mathbb{R} \times Osc^{k\alpha}(M)$. A system of local coordinates on $J^{\alpha k}(\mathbb{R},\ M)$ will be denoted by $(t, x, y^{(\alpha)}, y^{(2\alpha)}, ..., y^{(k\alpha)})$. Consider the projections $\pi_0^{\alpha k} : J^{\alpha k}(\mathbb{R},\ M) \to M$ defined by

$$\pi_0^{\alpha k}(t, x, y^{(\alpha)}, ..., y^{(\alpha k)}) = x. \tag{28}$$

Let $U,\ U' \subset M$ be two charts on M with $U \cap U' \neq \emptyset$, $(\pi_0^\alpha)^{-1}(U),\ (\pi_0^\alpha)^{-1}(U') \subset J^\alpha(\mathbb{R},\ M)$ the corresponding charts on $J^\alpha(\mathbb{R},\ M)$ and, respectively, the corresponding coordinates (x^i), $(\bar{x}^i)$ and $(t, x^i, y^{i(\alpha)})$, $(t, \bar{x}^i, \bar{y}^{i(\alpha)})$. The transformations of coordinates are given by

$$\begin{aligned} &\bar{x}^i = \bar{x}^i(x^1, ..., x^n) \\ &\bar{y}^{i(\alpha)} = \overset{\alpha}{J}(x, \bar{x}) y^{i(\alpha)}. \end{aligned} \tag{29}$$

Consider the functions $(t)^\alpha$, $(x^i)^\alpha$, $(y^{i(\alpha)})^\alpha \in \mathcal{F}((\pi_0^\alpha)^{-1}(U))$, the 1-forms $\frac{1}{\Gamma(1+\alpha)} d(t)^\alpha$, $\frac{1}{\Gamma(1+\alpha)} d(x^i)^\alpha$, $\frac{1}{\Gamma(1+\alpha)} d(y^{i(\alpha)})^\alpha \in \mathcal{D}^1((\pi_0^\alpha)^{-1}(U))$ and the operators D_t^α, $D_{x^i}^\alpha$, $D_{y^{i(\alpha)}}^\alpha$ on $(\pi_0^\alpha)^{-1}(U)$, $i = \overline{1, n}$. The following relations hold:

$$\begin{aligned} &D_t^\alpha \left(\tfrac{1}{\Gamma(1+\alpha)} t^\alpha\right) = 1, \quad D_{x^i}^\alpha \left(\tfrac{1}{\Gamma(1+\alpha)} (x^j)^\alpha\right) = \delta_i^j, \\ &D_{y^{i(\alpha)}}^\alpha \left(\tfrac{1}{\Gamma(1+\alpha)} (y^{j(\alpha)})^\alpha\right) = \delta_i^j, \quad \tfrac{1}{\Gamma(1+\alpha)} d(t^\alpha)(D_t^\alpha) = 1, \\ &\tfrac{1}{\Gamma(1+\alpha)} d(x^i)^\alpha (D_{x^j}^\alpha) = \delta_j^i, \quad \tfrac{1}{\Gamma(1+\alpha)} d(y^{i(\alpha)})^\alpha (D_{y^{j(\alpha)}}^\alpha) = \delta_j^i. \end{aligned} \tag{30}$$

On $J^\alpha(\mathbb{R},\ M)$ we may define the canonical structures

$$\begin{aligned} &\overset{\alpha}{\theta}_1 = d(t^\alpha) \otimes (D_t^\alpha + y^{i(\alpha)} D_{x^i}^\alpha) \\ &\overset{\alpha}{\theta}_2 = \overset{\alpha}{\theta^i} \otimes D_{x^i}^\alpha, \quad \overset{\alpha}{\theta^i} = \tfrac{1}{\Gamma(1+\alpha)} (d(x^i)^\alpha - y^{i(\alpha)} d(t)^\alpha) \\ &\overset{\alpha}{S} = \overset{\alpha}{\theta^i} \otimes D_{y^{i(\alpha)}}^\alpha \\ &\overset{\alpha}{V}_i = D_{y^{i(\alpha)}}^\alpha. \end{aligned} \tag{31}$$

Using (29) it is easy to show that the structures (31) have geometrical character. The space of the operators generated by the operators $\{D_t^\alpha, D_{x^i}^\alpha, D_{y^{i(\alpha)}}^\alpha\}$, $i = \overline{1,\ n}$, will be denoted by $\chi^\alpha((\pi_0^\alpha)^{-1}(U))$. For $\alpha \to 1$ the space of these operators represents the space of the vector fields on $\pi_0^{-1}(U)$.

A vector field $\overset{\alpha}{\Gamma} \in \chi^\alpha((\pi_0^\alpha)^{-1}(U))$ is called *FODE* (fractional ordinary differential equation) iff

$$\begin{aligned} &d(t)^\alpha(\overset{\alpha}{\Gamma}) = 1 \\ &\overset{\alpha}{\theta^i}(\overset{\alpha}{\Gamma}) = 0, \end{aligned} \tag{32}$$

for $i = \overline{1, n}$. In local coordinates *FODE* is given by

$$\overset{\alpha}{\Gamma} = D_t^\alpha + y^{i(\alpha)} D_{x^i}^\alpha + F^i D_{y^{i(\alpha)}}^\alpha, \tag{33}$$

where $F^i \in C^\infty((\pi_0^\alpha)^{-1}(U))$, $i = \overline{1,n}$. The integral curves of the field $FODE$ are the solutions of the fractional differential equation (EDF)

$$D_t^{2\alpha} x^i(t) = F^i(t, x(t), D_t^\alpha x(t)), \qquad i = \overline{1,n}. \tag{34}$$

The fractional dynamical connection on $J^\alpha(\mathbb{R},\ M)$ is defined by the fractional tensor fields $\overset{\alpha}{H}$ of type $(1,1)$ which satisfy the conditions

$$\begin{aligned} &\overset{\alpha}{\theta}_1 \circ \overset{\alpha}{H} = 0 \\ &\overset{\alpha}{\theta}_2 \circ \overset{\alpha}{H} = \overset{\alpha}{\theta}_2 \\ &\overset{\alpha}{H}\Big|_{\overset{\alpha}{V}} = -id\Big|_{\overset{\alpha}{V}}, \end{aligned} \tag{35}$$

where $\overset{\alpha}{V}$ is formed by operators generated by $\{D^\alpha_{y^{i(\alpha)}}\}_{i=\overline{1,n}}$. In the chart $(\pi_0^\alpha)^{-1}(U)$ the fractional tensor field $\overset{\alpha}{H}$ has the expression

$$\begin{aligned} \overset{\alpha}{H} = &(\overset{1}{H}\, d(t)^\alpha + \overset{2}{H}{}^i_j\, d(x^i)^\alpha + \overset{3}{H}_i\, d(y^{i(\alpha)})^\alpha) \otimes D_t^\alpha + \\ &(\overset{4}{H}{}^i_j (dt)^\alpha + \overset{5}{H}{}^i_j\, d(x^j)^\alpha + \overset{6}{H}{}^i_j\, d(y^{i(\alpha)})^\alpha) \otimes D^\alpha_{x^i} + \\ &(\overset{7}{H}{}^i\, d(t)^\alpha + \overset{8}{H}{}^i_j\, d(x^j)^\alpha + \overset{9}{H}{}^i_j\, d(y^{i(\alpha)})^\alpha) \otimes D^\alpha_{y^{i(\alpha)}}. \end{aligned} \tag{36}$$

The tensor field $\overset{\alpha}{H}$ has a geometrical character, fact which results by using the relations (29), and is called a d^α-tensor field. Using the relations (30) and (31) we get

Proposition 4. *a) The fractional dynamical connection $\overset{\alpha}{H}$, in the chart $(\pi_0^\alpha)^{-1}(U)$, is given by*

$$\begin{aligned} \overset{\alpha}{H} = &\tfrac{1}{\Gamma(1+\alpha)}[(-y^{i(\alpha)} D^\alpha_{x^i} + H^i D^\alpha_{y^{i(\alpha)}}) \otimes d(t)^\alpha + \\ &(D^\alpha_{x^i} + H^j_i D^\alpha_{y^{j(\alpha)}}) \otimes d(x^i)^\alpha - D^\alpha_{y^{i(\alpha)}} \otimes d(y^{i(\alpha)})^\alpha]. \end{aligned} \tag{37}$$

b) The fractional dynamical connection $\overset{\alpha}{H}$ defines a $f(3,-1)$ fractional structure on $J^\alpha(\mathbb{R},\ M)$, i.e., $\left(\overset{\alpha}{H}\right)^3 = \overset{\alpha}{H}$.
c) The fractional tensor fields $\overset{\alpha}{l}$ and $\overset{\alpha}{m}$ which are defined by

$$\begin{aligned} &\overset{\alpha}{l} = \overset{\alpha}{H} \circ \overset{\alpha}{H} \\ &\overset{\alpha}{m} = -\overset{\alpha}{H} \circ \overset{\alpha}{H} + I, \end{aligned} \tag{38}$$

where I is the identity map, satisfy the relations

$$\begin{aligned}
&\overset{\alpha}{l} \circ \overset{\alpha}{l} = \overset{\alpha}{l}, \ \ \overset{\alpha}{m} \circ \overset{\alpha}{m} = \overset{\alpha}{m} \circ \overset{\alpha}{l}, \ \ \overset{\alpha}{l} + \overset{\alpha}{m} = I \\
&\overset{\alpha}{l}(D_t^\alpha) = -y^{i(\alpha)} D_{x^i} - (y^{i(\alpha)} \overset{\alpha}{H}{}_i^j + \overset{\alpha}{H}{}^j) D_{y^{i(\alpha)}} \\
&\overset{\alpha}{l}(D_{x^i}^\alpha) = D_{x^i}^\alpha, \ \ \overset{\alpha}{l}(D_{y^{i(\alpha)}}^\alpha) = D_{y^{i(\alpha)}}^\alpha \\
&\overset{\alpha}{m}(D_t^\alpha) = D_t^\alpha + y^{i(\alpha)} D_{x^i}^\alpha + (y^{i(\alpha)} \overset{\alpha}{H}{}_i^j + \overset{\alpha}{H}{}^j) D_{y^{i(\alpha)}}^\alpha \\
&\overset{\alpha}{m}(D_{x^i}^\alpha) = 0, \ \ \overset{\alpha}{m}(D_{y^{i(\alpha)}}^\alpha) = 0.
\end{aligned} \tag{39}$$

d) The fractional vector field $\overset{\alpha}{\Gamma} \in \chi^\alpha(J^\alpha(\mathbb{R}, \ M))$ given by

$$\overset{\alpha}{\Gamma} = \overset{\alpha}{m}(D_t^\alpha) = D_t^\alpha + y^{i(\alpha)} D_{x^i}^\alpha + (y^{i(\alpha)} \overset{\alpha}{H}{}_i^j + \overset{\alpha}{H}{}^j) D_{y^{j(\alpha)}}^\alpha \tag{40}$$

defines a field FODE associated to the fractional dynamical connection. The integral curves are the solutions of the EDF

$$D_t^{2\alpha} x^i(t) = D_t^\alpha x^i(t) \overset{\alpha}{H}{}_i^j + \Gamma(1+\alpha) \overset{\alpha}{H}{}^j \tag{41}$$

where $\overset{\alpha}{H}{}_i^j$ and $\overset{\alpha}{H}{}^j$ are functions of $(t, \ x(t), \ y^{(\alpha)}(t))$.

Let $L \in C^\infty(J^\alpha(\mathbb{R}, \ M))$ be a fractional Lagrange function. By definition, the Cartan fractional 1-form is the 1-form $\overset{\alpha}{\theta}_L$ given by

$$\overset{\alpha}{\theta}_L = L d(t)^\alpha + \overset{\alpha}{S}(L). \tag{42}$$

We call the Cartan fractional 2-form, the 2-form $\overset{\alpha}{\omega}_L$ given by

$$\overset{\alpha}{\omega}_L = d^\alpha \overset{\alpha}{\theta}_L \tag{43}$$

where d^α is the fractional exterior differential:

$$d^\alpha = d(t)^\alpha D_t^\alpha + d(x^i)^\alpha D_{x^i}^\alpha + d(y^{i(\alpha)})^\alpha D_{y^{i(\alpha)}}^\alpha. \tag{44}$$

In the chart $(\pi_0^\alpha)^{-1}(U)$, $\overset{\alpha}{\theta}_L$ and $\overset{\alpha}{\omega}_L$ are given by

$$\begin{aligned}
\overset{\alpha}{\theta}_L &= (L - \tfrac{1}{\Gamma(1+\alpha)} y^{i(\alpha)} D_{y^{i(\alpha)}}^\alpha (L) d(t)^\alpha + \tfrac{1}{\Gamma(1+\alpha)} D_{y^{i(\alpha)}}^\alpha (L) d(x^i)^\alpha \\
\overset{\alpha}{\omega}_L &= A_i d(t)^\alpha \wedge d(x^i)^\alpha + B_i d(t^\alpha) \wedge d(y^{i(\alpha)})^\alpha + \\
&\quad A_{ij} d(x^i)^\alpha \wedge d(x^j)^\alpha + B_{ij} d(x^i)^\alpha \wedge d(y^{j(\alpha)})^\alpha,
\end{aligned} \tag{45}$$

where

$$\begin{aligned}
&A_i = \tfrac{1}{\Gamma(1+\alpha)} D_t^\alpha D_{y^{i(\alpha)}}^\alpha(L) + \tfrac{1}{\Gamma(1+\alpha)} y^{j(\alpha)} D_{x^i}^\alpha D_{y^{j(\alpha)}}^\alpha(L) - D_{x^i}^\alpha(L)\\
&B_i = \tfrac{1}{\Gamma(1+\alpha)} D_{y^{i(\alpha)}}^\alpha(y^{j(\alpha)} D_j^\alpha(\alpha)(L))\\
&A_{ij} = D_{x^i}^\alpha D_{y^{i(\alpha)}}^\alpha(L), \quad B_{ij} = -D_{y^{j(\alpha)}}^\alpha D_{y^{i(\alpha)}}^\alpha(L).
\end{aligned} \tag{46}$$

Proposition 5. *If L is regular (i.e.,* $\det\left(\frac{\partial^2 L}{\partial y^{i(\alpha)}\partial y^{j(\alpha)}}\right) \neq 0$*) then there exists a fractional field FODE* $\overset{\alpha}{\Gamma}_L$ *such that* $i_{\overset{\alpha}{\Gamma}_L} \overset{\alpha}{\omega}_L = 0$*. In the chart* $(\pi_0^\alpha)^{-1}(U)$ *we have*

$$\overset{\alpha}{\Gamma}_L = D_t^\alpha + y^{i(\alpha)} D_{x^i} + \overset{\alpha}{M}{}^i D_{y^{i(\alpha)}}^\alpha, \tag{47}$$

where

$$\begin{aligned}
&\overset{\alpha}{M}{}^i = g^{ik}(D_k^\alpha(L) - d_t^\alpha(\tfrac{\partial^\alpha L}{\partial y^{k(\alpha)}})\\
&d_t^\alpha = D_t^\alpha + y^{i(\alpha)} D_{x^i}^\alpha\\
&(g^{ik}) = (D_{y^{i(\alpha)}}^\alpha D_{y^{k(\alpha)}}^\alpha(L))^{-1}.
\end{aligned} \tag{48}$$

An important structure on $J^\alpha(\mathbb{R},\ M)$ is described by the fractional Euler-Lagrange equations. Let $c : t \in [0,1] \to (x^i(t)) \in M$ be a parameterized curve, such that $Imc \subset U \subset M$. The extension of the curve c to $J^\alpha(\mathbb{R},\ M)$ is the curve $c^\alpha : t \in [0,1] \to (t, x^i(t), y^{i(\alpha)}(t)) \in J^\alpha(\mathbb{R},\ M)$. Consider $L \in C^\infty(J^\alpha(\mathbb{R},\ M))$. The action of L along the curve c^α is defined by

$$\mathcal{A}(c^\alpha) = \int_0^1 L(t, x(t), y^\alpha(t))dt. \tag{49}$$

Let $c_\varepsilon : t \in [0,1] \to (x^i(t,\varepsilon)) \in M$ be a family of curves, where ε is sufficiently small so that $Imc_\varepsilon \subset U$, $c_0(t) = c(t)$, $D_\varepsilon^\alpha c_\varepsilon(0) = D_\varepsilon^\alpha c_\varepsilon(1) = 0$. The action of L along the curves c_ε is

$$\mathcal{A}(c_\varepsilon^\alpha) = \int_0^1 L(t, x(t,\varepsilon), y^\alpha(t,\varepsilon))dt, \tag{50}$$

where $y^{i(\alpha)}(t,\varepsilon) = \frac{1}{\Gamma(1+\alpha)} D_t^\alpha x^i(t,\varepsilon)$. The action (50) has a fractional extremal value if

$$D_\varepsilon^\alpha \mathcal{A}(c_\varepsilon^\alpha)\,|_{\varepsilon=0} = 0. \tag{51}$$

The action (50) has an extremal value if

$$D_\varepsilon^1 \mathcal{A}(c_\varepsilon^\alpha)\,|_{\varepsilon=0} = 0. \tag{52}$$

Using the properties of the fractional derivative we obtain

Proposition 6. *a) A necessary condition for the action* (50) *to reach a fractional extremal value is that* $c(t)$ *satisfies the fractional Euler-Lagrange equations*

$$\begin{aligned} &D^{\alpha}_{x^i}L - d^{2\alpha}_t(D^{\alpha}_{y^{i(\alpha)}}L) = 0 \\ &d^{\alpha}_t = D^{\alpha}_t + y^{i(\alpha)}D^{\alpha}_{x^i} + y^{i(2\alpha)}D^{\alpha}_{y^{i(\alpha)}}, \end{aligned} \tag{53}$$

where $i = \overline{1,n}$.
b) A necessary condition for the action (50) *to reach an extremal value is that* $c(t)$ *satisfies the Euler-Lagrange equations*

$$\begin{aligned} &D^{1}_{x^i}L - d^{2}_t(D^{1}_{y^{i(\alpha)}}L) = 0 \\ &d^{2}_t = D^{\alpha}_t + y^{i(\alpha)}D^{1}_{x^i} + y^{i(2\alpha)}D^{1}_{y^{i(\alpha)}}, \end{aligned} \tag{54}$$

where $i = \overline{1,n}$.

The equations (53) may be written in the form

$$D^{\alpha}_{x^i}L - d^{\alpha}_t(D^{\alpha}_{y^{i(\alpha)}}L) - y^{j(2\alpha)}D^{\alpha}_{y^{j(\alpha)}}(D^{\alpha}_{y^{i(\alpha)}}L) = 0, \tag{55}$$

for $i = \overline{1,n}$. The equations (54) may be written as

$$\frac{\partial L}{\partial x^i} - d^{\alpha}_t\left(\frac{\partial L}{\partial y^{i(\alpha)}}\right) - y^{j(2\alpha)}\frac{\partial^2 L}{\partial y^{i(\alpha)}\partial y^{j(\alpha)}} = 0, \tag{56}$$

where $i = \overline{1,n}$. Let us denote by

$$g^{\alpha}_{ij} = D^{\alpha}_{y^{i(\alpha)}}(D^{\alpha}_{y^{j(\alpha)}}L), \tag{57}$$

and by $\left(\overset{\alpha}{g}{}^{ik}\right) = \left(g^{\alpha}_{ij}\right)^{-1}$, if $\det(g^{\alpha}_{ij}) \neq 0$. From (55) and from Proposition 5, we get the fractional field $FODE$ $\overset{\alpha}{\Gamma}_L$ associated to L.
Let $c : t \in [0,1] \to (x^i(t)) \subset U$ be a parameterized curve. The extension of c to $J^{\alpha k}(\mathbb{R},\ M)$ is the curve $c^{\alpha k} : t \in [0,1] \to (t, x^i(t), y^{\alpha a}(t)) \in J^{\alpha k}(\mathbb{R},\ M)$, $a = \overline{1,k}$. Let $L : J^{\alpha k}(\mathbb{R},\ M) \to \mathbb{R}$ be a Lagrange function. The action of L along the curve $c^{\alpha k}$ is

$$\mathcal{A}(c^{\alpha k}) = \int_0^1 L(t, x(t), y^{\alpha a}(t))dt. \tag{58}$$

Let $c_\varepsilon : t \in [0,1] \to (x^i(t,\varepsilon)) \in M$ be a family of curves, where the absolute value of ε is sufficiently small so that $Imc_\varepsilon \subset U \subset M$, $c_0(t) = c(t)$, $D^{\alpha}_{\varepsilon}c(\varepsilon)\,|_{\varepsilon=0} = D^{\alpha}_{\varepsilon}c(\varepsilon)\,|_{\varepsilon=1} = 0$. The action of L on the curve c_ε is given by

$$\mathcal{A}(c^{\alpha k}_\varepsilon) = \int_0^1 L(t, x(t,\varepsilon), y^{\alpha a}(t,\varepsilon))dt \tag{59}$$

where $y^{i(\alpha a)}(t,\varepsilon) = \frac{1}{\Gamma(1+\alpha a)} D_t^{\alpha a} x^i(t,\varepsilon)$, $a = \overline{1,k}$. The action (59) has a fractional extremal value if

$$D_\varepsilon^\alpha(\mathcal{A}(c_\varepsilon^{\alpha k}))\,|_{\varepsilon=0} = 0. \tag{60}$$

The action (59) has an extremal value if

$$D_\varepsilon^1(\mathcal{A}(c_\varepsilon^{\alpha k}))\,|_{\varepsilon=0} = 0. \tag{61}$$

Proposition 7. *a) A necessary condition for the action* (58) *to reach a fractional extremal value is that* $c(t)$ *satisfies the fractional Euler-Lagrange equations*

$$D_{x^i}^\alpha L + \sum_{a=1}^{k} (-1)^a d_t^{\alpha a}(D_{y^{i(\alpha a)}}^\alpha L) = 0, \tag{62}$$

where

$$d_t^{\alpha a} = D_t^\alpha + y^{i(\alpha)} D_{x^i}^\alpha + y^{i(2\alpha)} D_{y^{i(\alpha)}}^\alpha + ... + y^{i(\alpha a)} D_{y^{i(\alpha(a-1))}}^\alpha, \tag{63}$$

and $i = \overline{1,n}$.
b) A necessary condition that the action (58) *reaches an extremal value is that* $c(t)$ *satisfies the Euler-Lagrange equations*

$$\frac{\partial L}{\partial x^i} + \sum_{a=1}^{k} (-1)^a d_t^{a}(D_{y^{i(\alpha a)}}^\alpha L) = 0, \tag{64}$$

where

$$d_t^{a} = D_t^1 + y^{i(\alpha)} D_{x^i}^1 + ... + y^{i(\alpha a)} D_{y^{i(\alpha(a-1))}}^1. \tag{65}$$

Example. Consider the fractional differential equation

$$\begin{aligned}&\tfrac{c\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha}(t) f(t) + a_1\Gamma(1+2\alpha) y^{(2\alpha)} + \\ &a_2\Gamma(1+3\alpha) y^{(3\alpha)} = 0.\end{aligned} \tag{66}$$

The equation (66) is the fractional Euler-Lagrange equation (62) for the function

$$\begin{aligned}L = &\tfrac{c}{1+\gamma-\alpha} x^{\gamma} - a_1\Gamma(1+2\alpha)(y^\alpha)^\alpha + \\ &a_2\Gamma(1+3\alpha)(y^{2\alpha})^\alpha.\end{aligned}$$

The equation (66) is the fractional Euler-Lagrange equation (64) for the function

$$L = \frac{c\Gamma(1+\gamma) x^{\gamma-\alpha+1}}{\Gamma(1+\gamma-\alpha)^{(1+\gamma-\alpha)}} f - \frac{a_1}{2}\Gamma(1+2\alpha)(y^\alpha)^2 + \frac{a_2}{2}\Gamma(1+3\alpha)(y^{2\alpha})^2.$$

5 Examples and applications

1. **The nonhomogeneous Bagley-Torvik equation**

The dynamics of a flat rigid body embedded in a Newton fluid is described by the equation

$$aD_t^2x(t) + bD_t^{3/2}x(t) + cx(t) - f(t) = 0, \tag{67}$$

where $a, b, c \in \mathbb{R}$ and the initial conditions are $x(0) = 0$, $D_t^1x(0) = 0$. The equation (67) is a fractional differential equation on the bundle $J^\alpha(\mathbb{R}, \mathbb{R})$ for $\alpha = \frac{1}{4}$. Indeed, let's consider the fractional differential equation

$$aD_t^{8\alpha}x(t) + bD_t^{6\alpha}x(t) + cx(t) - f(t) = 0, \tag{68}$$

with $\alpha > 0$. For $\alpha = \frac{1}{4}$ the equation (68) reduces to (67). With the notations (15), the equation (68) becomes

$$a\Gamma(1 + 8\alpha)y^{(8\alpha)}(t) + b\Gamma(1 + 6\alpha)y^{(6\alpha)}(t) + cx(t) - f(t) = 0. \tag{69}$$

On the bundle $J^{4\alpha}(\mathbb{R}, \mathbb{R})$ let us consider the Lagrange function

$$\begin{aligned} &L(t, x, y^{(3\alpha)}, y^{(4\alpha)}) = \tfrac{1}{2}cx^2 - fx - \tfrac{b}{2}\Gamma(1 + 6\alpha)(y^{(3\alpha)})^2+ \\ &\tfrac{a}{2}\Gamma(1 + 8\alpha)(y^{(4\alpha)})^2. \end{aligned} \tag{70}$$

Using the relation (65), the Euler-Lagrange equation for (70) is

$$\begin{aligned} &D_x^1L - D_t^{3\alpha}(D^1_{y^{(3\alpha)}}L) + D_t^{4\alpha}(D^1_{y^{(4\alpha)}}L) = \\ &cx - f + b\Gamma(1 + 6\alpha)D_t^{3\alpha}y^{(3\alpha)} + a\Gamma(1 + 8\alpha)D_t^{4\alpha}y^{(4\alpha)} = \\ &cx - f + b\Gamma(1 + 6\alpha)y^{(6\alpha)} + a\Gamma(1 + 8\alpha)y^{(8\alpha)} = 0. \end{aligned} \tag{71}$$

Proposition 8. *The equation* (67) *represents the Euler-Lagrange equation on the bundle* $J^{4\alpha}(\mathbb{R}, \mathbb{R})$ *for* $\alpha = \frac{1}{4}$, *with the Lagrange function given by*

$$\begin{aligned} &L(t, x, y^{(3/2)}, y^{(2)}) = \tfrac{1}{2}cx^2 - fx - \tfrac{b}{2}\Gamma(5/2)(y^{(3/2)})^2+ \\ &\tfrac{a}{2}\Gamma(3)(y^{(2)})^2. \end{aligned} \tag{72}$$

2. **Differential equations of order one, two and three which admit fractional Lagrangians**

The following differential equations don't have classical Lagrangians such that the Euler-Lagrange equation represents the given equation:

$$\dot{x}(t) + V_1(t, x) = 0, \quad V_1(t, x) = \frac{\partial U_1(t, x)}{\partial x}, \tag{73}$$

$$\ddot{x}(t) + a_1\dot{x}(t) + V_2(t,x) = 0, \ \ V_2(t,x) = \frac{\partial U_2(t,x)}{\partial x}, \tag{74}$$

$$\dddot{x}(t) + a_2\ddot{x}(t) + a_1\dot{x}(t) + V_3(t,x) = 0, \ \ V_3(t,x) = \frac{\partial U_3(t,x)}{\partial x}. \tag{75}$$

Let us associate the fractional equations from below to the equations (73), (74) and (75), respectively:

$$D_t^{2\alpha}x(t) + V_1(t,x) = 0, \tag{76}$$

$$D_t^{4\alpha}x(t) + a_1 D_t^{2\alpha}x(t) + V_2(t,x) = 0, \tag{77}$$

$$D_t^{6\alpha}x(t) + a_2 D_t^{4\alpha}x(t) + a_1 D_t^{2\alpha}x(t) + V_3(t,x) = 0. \tag{78}$$

Proposition 9. *a) Let $J^\alpha(\mathbb{R},\ \mathbb{R}) \to \mathbb{R}$ be the fractional bundle and consider $L : J^\alpha(\mathbb{R},\ \mathbb{R}) \to \mathbb{R}$ given by*

$$L(t,x,y^{(\alpha)}) = U_1(t,x) - \frac{1}{2}\Gamma(1+2\alpha)(y^\alpha)^2. \tag{79}$$

The Euler-Lagrange equation of (79) *is*

$$\begin{aligned} &\tfrac{\partial L}{\partial x} - D_t^\alpha\left(\tfrac{\partial L}{\partial y^\alpha}\right) = \tfrac{\partial U_1(t,x)}{\partial x} + \Gamma(1+2\alpha)y^{(2\alpha)} = \\ &V_1(t,x) + D_t^{2\alpha}x(t) = 0. \end{aligned} \tag{80}$$

b) Let $J^{2\alpha}(\mathbb{R},\ \mathbb{R}) \to \mathbb{R}$ be the fractional bundle and the Lagrangian L : $J^{2\alpha}(\mathbb{R},\ \mathbb{R}) \to \mathbb{R}$ given by

$$\begin{aligned} &L(t,x,y^{(\alpha)},y^{(2\alpha)}) = U_2(t,x) - \tfrac{1}{2}a_1\Gamma(1+2\alpha)(y^\alpha)^2 + \\ &\tfrac{1}{2}\Gamma(1+4\alpha)(y^{(2\alpha)})^2. \end{aligned} \tag{81}$$

The Euler-Lagrange equation of (81) *is*

$$\begin{aligned} &\tfrac{\partial L}{\partial x} - D_t^\alpha\left(\tfrac{\partial L}{\partial y^\alpha}\right) + D_t^{2\alpha}\left(\tfrac{\partial L}{\partial y^{2\alpha}}\right) = \\ &V_2(t,x) + a_1\Gamma(1+2\alpha)y^{(2\alpha)} + \\ &a_2\Gamma(1+4\alpha)y^{(4\alpha)} = \\ &V_2(t,x) + a_1 D_t^{2\alpha}x(t) + D_t^{4\alpha}x(t) = 0. \end{aligned} \tag{82}$$

c) Let $J^{3\alpha}(\mathbb{R},\ \mathbb{R}) \to \mathbb{R}$ be the fractional bundle and $L : J^{3\alpha}(\mathbb{R},\ \mathbb{R}) \to \mathbb{R}$ given by

$$\begin{aligned} &L(t,x,y^{(\alpha)},y^{(2\alpha)},y^{(3\alpha)}) = V_3(t,x) - \tfrac{a_1}{2}\Gamma(1+2\alpha)(y^{(\alpha)})^2 + \\ &\tfrac{a_2}{2}\Gamma(1+4\alpha)(y^{(2\alpha)})^2 - \tfrac{1}{2}\Gamma(1+6\alpha)(y^{(3\alpha)})^2. \end{aligned} \tag{83}$$

The Euler-Lagrange equation of (83) *is*

$$\begin{aligned}&\tfrac{\partial L}{\partial x}-D_t^{\alpha}\left(\tfrac{\partial L}{\partial y^{\alpha}}\right)+D_t^{2\alpha}\left(\tfrac{\partial L}{\partial y^{(2\alpha)}}\right)-D_t^{3\alpha}\left(\tfrac{\partial L}{\partial y^{(3\alpha)}}\right)=V_3(t,x)+\\&a_1\Gamma(1+2\alpha)y^{(2\alpha)}+a_2\Gamma(1+4\alpha)y^{(4\alpha)}+\Gamma(1+6\alpha)y^{(6\alpha)}=\\&V_3(t,x)+a_1D_t^{2\alpha}x(t)+a_2D_t^{4\alpha}x(t)+D_t^{6\alpha}x(t)=0.\end{aligned}\tag{84}$$

d) For $\alpha=\frac{1}{2}$ *we obtain the fractional Lagrangians that describe the equations* (73)*,* (74)*,* (75)*, respectively*

$$\begin{aligned}&L(t,x,y^{(1/2)})=U_1(t,x)-\tfrac{1}{2}\Gamma(2)(y^{(1/2)})^2\\&L(t,x,y^{(1/2)},y^{(1)})=U_2(t,x)-\tfrac{1}{2}a_1\Gamma(2)(y^{(1/2)})^2+\tfrac{1}{2}\Gamma(3)(y^{(1)})^2\\&L(t,x,y^{(1/2)},y^{(1)},y^{(3/2)})=U_3(t,x)-\tfrac{a_1}{2}\Gamma(2)(y^{(1/2)})^2+\\&\tfrac{a_2}{2}\Gamma(2)(y^{(1)})^2-\tfrac{1}{2}\Gamma(4)(y^{(3/2)})^2.\end{aligned}\tag{85}$$

In the category of the equations (74) and (75) there are:
a) the nonhomogeneous classical friction equation

$$m\ddot{x}(t)+\gamma\dot{x}(t)-\frac{\partial U(t,x)}{\partial x}=0,\tag{86}$$

b) the nonhomogeneous model of Phillips [8]

$$\ddot{x}(t)+a_1\dot{x}(t)+b_1x(t)+f(t)=0,\tag{87}$$

c) the nonhomogeneous business cycle with innovation [8]

$$\dddot{y}(t)+a_2\ddot{y}(t)+a_1\dot{y}(t)+b_1x(t)+f(t)=0.\tag{88}$$

Conclusions

The paper presents the main differentiable structures on $J^{\alpha}(\mathbb{R},\ M)$, in order to describe fractional differential equations and ordinary differential equations, using Lagrange functions defined on $J^{\alpha}(\mathbb{R},\ M)$.

With the help of the methods shown, there may be analyzed other models, such as those found in [4] and [11].

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